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Assignment Games satisfy the CoMa-property^{xy}

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Abstract: A balanced game satisfies the CoMa-property if and only if the extreme points of its core are marginal vectors. In this note we prove that assignment games (cf. Shapley and Shubik (1972)) satisfy the CoMa-property.

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1 Introduction

A balanced game satisfies the CoMa-property if and only if the extreme points of its core are marginal vectors. Hence, the core of a game that satisfies the CoMa-property is the convex hull of the marginal vectors that are in the core.

A well-known class of games that satisfy the CoMa-property is the class of convex games: the core of a convex game is the convex hull of all marginal vectors (cf. Shapley (1971) and Ichiishi (1981)). A non-convex class of games that satisfy the CoMa-property is the class of information games (cf. Kuipers (1993)), which is a subclass of minimum cost spanning tree games (cf. Granot and Huberman (1981)). In this note we prove that assignment games (cf. Shapley and Shubik (1972)) satisfy the CoMa-property.

2 Preliminaries

In this section we recall some game theoretic notions and introduce the CoMa-property.

A *cooperative game with transferable utilities* is a pair $(P; v)$ where P is a finite set of players and $v : 2^P \rightarrow \mathbb{R}$ is a map that assigns to each $S \subseteq 2^P$ a real number $v(S)$, such that $v(\emptyset) = 0$. Here, 2^P is the collection of all subsets (coalitions) of P .

Assignment games, introduced by Shapley and Shubik (1972), arise from bipartite matching situations. Let M and N be two disjoint sets. For each $i \in M$ and $j \in N$ the value of a matched pair $(i; j)$ is $a_{ij} \geq 0$. From this situation an assignment game is defined in the following way. On the player set $M \cup N$, the worth of coalition $S \subseteq M \cup N$, $S \cap M \neq \emptyset$, $S \cap N \neq \emptyset$ is defined to be the maximum that S can achieve by making suitable pairs from its members. If $S = \emptyset$ or $S \cap M = \emptyset$ or $S \cap N = \emptyset$; no suitable pairs can be made and therefore the worth in this situation is 0. Formally, an *assignment game* $(M \cup N; w)$ is defined by

$$w(S \subseteq M \cup N) := \max_{\mu \in M(S; T)} \sum_{(i; j) \in \mu} a_{ij} \quad \text{for all } S \subseteq M \cup N,$$

where $M(S; T)$ denotes the set of matchings between S and T . A matching $\mu \in M(S; T)$ is called *optimal* for $S \subseteq M \cup N$ if $\sum_{(i; j) \in \mu} a_{ij} = w(S \subseteq M \cup N)$.

The core of a game $(P; v)$ consists of all vectors that distribute the gains $v(P)$ among the players in P in such a way that no subset of players can be better off by seceding from the rest of the players and act on their own behalf. Formally, the *core* of a game $(P; v)$ is defined by

$$\text{Core}(P; v) := \{x \in \mathbb{R}^P \mid x(S) \leq v(S) \text{ for all } S \subseteq P \text{ and } x(P) = v(P)\};$$

where $x(S) := \sum_{i \in S} x_i$. A game is *balanced* if and only if its core is non-empty (cf. Bondareva (1963) and Shapley (1967)). Shapley and Shubik (1972) showed that assignment games are balanced. A core allocation $x \in \text{Core}(M \cup N; w)$ of an assignment game $(M \cup N; w)$ will sometimes, for convenience, be denoted by $(u; v) \in \mathbb{R}^M \times \mathbb{R}^N$, where u and v are the vectors that correspond to the payoffs of the players in M and N , respectively.

Let $(P; v)$ be a game. Let $\pi(P)$ be the set of all orderings of P , i.e., bijections $\pi : P \rightarrow \{1, \dots, |P|\}$. For $\pi \in \pi(P)$, the *marginal vector* $m^\pi(v)$ is defined by

$$m_i^\pi(v) := v(\{j \in P \mid \pi(j) < \pi(i)\}) + v(\{j \in P \mid \pi(j) = \pi(i)\}) - v(\{j \in P \mid \pi(j) < \pi(i)\}) \quad \text{for all } i \in P.$$

Now, we are able to define the CoMa-property for a balanced game. A balanced game $(P; v)$ satisfies the *Core is convex hull of Marginals (CoMa-) property* if

$$\text{Core}(P; v) = \text{conv}\{m^j(v) \mid j \in (P) \text{ and } m^j(v) \in \text{Core}(P; v)\}.$$

3 The CoMa-property for assignment games

The main result of the note is formulated in Theorem 3.1.

Theorem 3.1 *Assignment games satisfy the CoMa-property.*

For the proof of Theorem 3.1 we need some lemmas.

Let $(M \sqcup N; w)$ be an assignment game. The following lemma, due to Shapley and Shubik (1972), gives a set of conditions that is necessary and sufficient for an allocation to be in the core.

Lemma 3.2 *Let $(M \sqcup N; w)$ be an assignment game and let π^1 be an optimal matching between M and N . Let $x = (u; v) \in \mathbb{R}^M \times \mathbb{R}^N$. Then, $x \in \text{Core}(M \sqcup N; w)$ if and only if the following four conditions are satisfied:*

- (i) $u_i + v_j = a_{ij}$ for all $(i; j) \in \pi^1$;
- (ii) $u_i + v_j \leq a_{ij}$ for all $i \in M; j \in N$; and $(i; j) \notin \pi^1$;
- (iii) $x_k = 0$ for all unmatched players k ;
- (iv) $x_k \geq 0$ for all matched players k .

Let $(M \sqcup N; w)$ be an assignment game. Let π^1 be an optimal matching between M and N . Given a core allocation $(u; v) \in \text{Core}(M \sqcup N; w)$, in the *tight graph* $G^w(u; v) = (V; E)$, the set of vertices V equals the player set $M \sqcup N$ and the edge set is defined by $E := \{(i; j) \mid i \in M; j \in N; \text{ and } u_i + v_j = a_{ij}\}$. In a tight graph we distinguish between two types of edges with respect to π^1 . All edges corresponding to π^1 are referred to as thick edges and all other edges are referred to as thin edges. Given a component of a tight graph we can construct a tree¹ that is a subgraph of the component, covers all vertices of the component, and contains all thick edges in the component. Such a tree we call a *tight tree*. Notice that a tight tree need not be uniquely determined by the tight graph. The following lemma establishes a relation between the extreme points of the core of an assignment game and the components of the corresponding tight graph.

Lemma 3.3 *Let $(M \sqcup N; w)$ be an assignment game and let π^1 be an optimal matching between M and N . Let $(u; v) \in \text{Core}(M \sqcup N; w)$. Then, $(u; v) \in \text{extfCore}(M \sqcup N; w)$ if and only if each component of the tight graph $G^w(u; v)$ contains at least one player with payoff equal to zero.*

Proof. First we show the ‘only if’-part. Let $(u; v) \in \text{Core}(M \sqcup N; w)$ and let C be a component of $G^w(u; v)$ in which the players are $S \sqcup T$ ($S \subseteq M; T \subseteq N$). Suppose that the restriction of

¹A tree is a connected graph without circuits.

$(u; v)$ to $(S; T)$, denoted by $(u; v)_{jS \sqcup T}$, has only positive elements. Then, by Lemma 3.2(iii), all players in $S \sqcup T$ are matched by π^1 . By definition of a component, all players in $S \sqcup T$ are matched within $S \sqcup T$. Hence, $|jSj| = |jTj|$. Then, by Lemma 3.2(i)-(iv), we have that for sufficiently small $\epsilon^2 > 0$ the vectors $y; z \in \mathbb{R}^M \times \mathbb{R}^N$ defined by $y_i := u_i + \epsilon^2$, $z_i := u_i - \epsilon^2$ for $i \in S$; $y_j := v_j - \epsilon^2$; $z_j := v_j + \epsilon^2$ for $j \in T$; $y_l := x_l = z_l$ for $l \in (M \sqcup N) \setminus (S \sqcup T)$, are both in $\text{Core}(M \sqcup N; w)$. Together with $\frac{1}{2}y + \frac{1}{2}z = x = (u; v)$ this implies that $(u; v) \notin \text{extfCore}(M \sqcup N; w)$, proving the ‘only if’-part.

To prove the ‘if’-part, we assume that each component of the tight graph $G^w(u; v)$ contains at least one player with payoff equal to zero. It is sufficient to show that the system

$$u(S) + v(T) \leq w(S \sqcup T) \quad \text{for all } S \sqcup M; T \sqcup N$$

contains $|jMj| + |jNj|$ tight equations that are linearly independent. Let $C_1; C_2; \dots; C_k$ be the components of the tight graph $G^w(u; v)$. Let $P(C_l)$ be the set of players corresponding to C_l for all $l = 1; \dots; k$. Each component C_l contains a tight tree. Then the system of equations generated by the edges of such a tree is a linearly independent system (cf. Chvátal (1983)). Hence, we have $\sum_{l=1}^k (|jP(C_l)j| - 1)$ linearly independent tight equations. Combining these equations with the tight equations generated by the players with zero payoff we obtain a system of $\sum_{l=1}^k (|jP(C_l)j|) = |jMj| + |jNj|$ linearly independent equations. \square

The following lemma provides the worth of some specific $(r \sqcup s)$ -path coalitions. For two players $r; s \in P$, $r \neq s$, that are in the same tight tree, an $(r \sqcup s)$ -path coalition consists of all players that are contained in the unique path between r and s , including r and s .

Lemma 3.4 *Let $(u; v)$ be an extreme point of the core of an assignment game $(M \sqcup N; w)$ and let π^1 be an optimal matching between M and N . Suppose S is an $(r \sqcup s)$ -path coalition in a tight tree of $G^w(u; v)$ for which vertex r corresponds to a player that has a payoff equal to zero in $(u; v)$. Then, $w(S) = \sum_{i \in S \setminus M} u_i + \sum_{j \in S \setminus N} v_j$:*

Proof. Let π^0 be the matching between $S \setminus M$ and $S \setminus N$ that 1) covers S if $|jSj|$ is even and $S \setminus r$ if $|jSj|$ is odd and 2) only consists of pairs that correspond to edges in the $(r \sqcup s)$ -path. Then from the definition of π^0 , the equalities $u_i + v_j = a_{ij}$ (for all $(i; j) \in \pi^0$), and the fact that the payoff corresponding to vertex r equals 0 it follows that

$$\sum_{i \in S \setminus M} u_i + \sum_{j \in S \setminus N} v_j = \sum_{(i; j) \in \pi^0} a_{ij} \quad (1)$$

From the definition of an assignment game and (1) we have that

$$w(S) \leq \sum_{i \in S \setminus M} u_i + \sum_{j \in S \setminus N} v_j \quad (2)$$

On the other hand, since $(u; v)$ is a core allocation, the reverse inequality of (2) holds, providing the desired equality. \square

Now, we can prove Theorem 3.1.

Proof of Theorem 3.1. Note that if a game arises from another game by adding null players only, then the ‘larger’ game satisfies the CoMa-property if and only if the ‘smaller’ one does. This assertion immediately follows from the well-known fact that the core of the larger game arises by adding 0 components for the null players to any core element of the smaller one. Therefore, we may restrict attention to assignment games $(M \sqcup N; w)$ with $|M| = |N|$.

Let $(M \sqcup N; w)$ be an assignment game with $|M| = |N|$. Take $x = (u; v) \in \text{extCore}(M \sqcup N; w)$. We have to show that there exists some ordering γ of the player set $M \sqcup N$ such that the corresponding marginal vector $m^\gamma(w)$ coincides with $(u; v)$. The proof consists of three parts. Let μ be an optimal matching between M and N that matches all players in $M \sqcup N$. Let $C_1; C_2; \dots; C_k$ be the components of the tight graph $G^w(u; v)$ and let $P(C_l)$ be the set of players corresponding to C_l for all $l = 1; \dots; k$.

Claim 1. Let $S \subseteq M \sqcup N$. If $x(S \setminus P(C_l)) = w(S \setminus P(C_l))$ for all $l = 1; \dots; k$, then $x(S) = w(S)$.

Proof. Note that

$$\begin{aligned} x(S) &= x(S \setminus P(C_1)) + \dots + x(S \setminus P(C_k)) \\ &= w(S \setminus P(C_1)) + \dots + w(S \setminus P(C_k)) \\ &= w(S); \end{aligned}$$

where the second equality follows from the assumption and the inequality from the fact that the merger of optimal matchings for $S \setminus P(C_1); S \setminus P(C_2); \dots; S \setminus P(C_k)$ gives a matching for S . On the other hand, since $x \in \text{Core}(M \sqcup N; w)$, we have that $x(S) \leq w(S)$. We conclude that $x(S) = w(S)$. \square

For a component C_l , a *tight sequence* $\gamma_l = S_0^l \subsetneq S_1^l \subsetneq S_2^l \subsetneq \dots \subsetneq S_{|P(C_l)|}^l = P(C_l)$ is a strictly increasing sequence of coalitions with $x(S_j^l) = w(S_j^l)$ for all $j = 1; \dots; |P(C_l)|$. Clearly, $|S_j^l \cap S_{j-1}^l| = 1$ for all $j = 1; \dots; |P(C_l)|$.

Claim 2. Suppose that for every component C_l there is a tight sequence $\gamma_l = S_0^l \subsetneq S_1^l \subsetneq S_2^l \subsetneq \dots \subsetneq S_{|P(C_l)|}^l = P(C_l)$. Define $\gamma : M \sqcup N \rightarrow \{1; \dots; |M| + |N|\}$ by $\gamma(i) := \sum_{l=1}^k |P(C_l)| + j$ if $i \in S_j^{q+1} \setminus S_{j-1}^{q+1}$ for $0 \leq q \leq k-1$ and $1 \leq j \leq |P(C_{q+1})|$. Then, $m^\gamma(w) = x$.

Proof. Let S_j be the set of the first j players in $M \sqcup N$ with respect to the ordering γ , i.e.,

$$S_j := \{i \in M \sqcup N \mid \gamma(i) \leq j\}.$$

From the definition of tight sequence and Claim 1 it follows that $x(S_j) = w(S_j)$ for all $j = 1; \dots; |M| + |N|$. Now, take $i \in M \sqcup N$. Then,

$$\begin{aligned} m_i^\gamma(w) &= w(S_{\gamma(i)}) - w(S_{\gamma(i)-1}) \\ &= x(S_{\gamma(i)}) - x(S_{\gamma(i)-1}) \\ &= x_i; \end{aligned}$$

where the first equality follows from the definition of a marginal vector and the second equality from $x(S_j) = w(S_j)$ for all $j = 1; \dots; jMj + jNj$. Hence, $m^w(w) = x$. \square

The theorem now follows from Claim 2 and Claim 3.

Claim 3. For every component C_l there is a tight sequence $\emptyset \subseteq S_1^l \subseteq S_2^l \subseteq \dots \subseteq S_{jP(C_l)}^l = P(C_l)$.

Proof. Let T_l be a tight tree of the component C_l . Since π^1 matches all players, it follows from the definition of a component that the vertices of C_l , and hence the vertices of T_l , form a set of matched pairs $(i; j) \in \pi^1$. Lemma 3.3 implies that there exists a vertex r in the tight tree T_l the payoff of which is 0. We take such a vertex r and we call it the root of the tree T_l . Now T_l is a rooted tree, i.e., a tree with a distinguished vertex – the root. Clearly, the root r determines a direction of the edges as follows. An edge $fa; bg$ in the rooted tree T_l is directed from vertex a to vertex b if a is on the unique path from r to b . The directed rooted tree T_l with root r is called r -tree.

Next, we label the vertices in the r -tree T_l by $1; 2; \dots; jP(C_l)j$ via the following procedure.

STEP 1: Give vertex r label 1. Set $a := r$ and $t := 1$.

STEP 2:

(i) If there exists a thin edge that connects a with an unlabeled vertex b , then give vertex b label $t + 1$, set $t := t + 1$ and $a := b$, and go to Step 3. Otherwise go to (ii).

(ii) If there exists a thick edge that connects a with an unlabeled vertex b , then give vertex b label $t + 1$, set $t := t + 1$ and $a := b$, and go to Step 3. Otherwise, scan vertex a , let b be the predecessor of a in the rooted tree T_l , set $a := b$, and go to (i).

STEP 3: If $t = jP(C_l)j$, then STOP. Otherwise go to Step 2.

Note that in every visit of Step 2 we either label or scan a vertex. Since every vertex gets labeled and scanned only once (except for the root and the vertex labeled last, which do not get scanned), the procedure ends after at most $2jP(C_l)j + 3$ visits of Step 2. Let S_j^l be the set of the first j labeled players in the procedure. Clearly, $\emptyset \subseteq S_1^l \subseteq S_2^l \subseteq \dots \subseteq S_{jP(C_l)}^l = P(C_l)$. So, we are done if we prove that

$$w(S_j^l) = x(S_j^l) \quad \text{for all } j = 1; \dots; jP(C_l)j. \quad (3)$$

Let player m be the player that is labeled last in S_j^l . Coalition S_j^l can be partitioned in $S_j^l(1)$ and $S_j^l(2)$, where $S_j^l(1)$ is the set of players on the unique path from r to m and $S_j^l(2)$ is the set of all other players in S_j^l . Then Lemma 3.4 implies that

$$w(S_j^l(1)) = x(S_j^l(1)): \quad (4)$$

Obviously, the proof is completed if $S_j^l(2) = \emptyset$. Hence, we may assume that $S_j^l(2) \neq \emptyset$.

We now show that μ^1 matches every player in $S_j^1(2)$ with another player in $S_j^1(2)$. Let $a \in S_j^1(1)$ and let $b \in S_j^1(2)$ be such that $(a; b)$ is an edge in the r -tree. Consider now the edge $(a; b)$ and the path from a to m . Since vertex b is visited before m is visited, it follows from Step 2(i) of the procedure that $(a; b)$ is a thin edge. Since μ^1 matches all players, it follows that μ^1 matches every player in $S_j^1(2)$ with another player in $S_j^1(2)$. This observation gives

$$w(S_j^1(2)) = x(S_j^1(2)); \quad (5)$$

since an optimal matching for $S_j^1(2)$ is provided by the thick edges in $S_j^1(2)$. Now, we have

$$\begin{aligned} w(S_j^1) &\leq w(S_j^1(1)) + w(S_j^1(2)) \\ &= x(S_j^1(1)) + x(S_j^1(2)) \\ &= x(S_j^1) \\ &\leq w(S_j^1); \end{aligned} \quad (6)$$

where the first inequality holds since the merger of optimal matchings of $S_j^1(1)$ and $S_j^1(2)$ gives a matching for S_j^1 , the first equality holds by (4) and (5), the second equality since $S_j^1(1)$ and $S_j^1(2)$ form a partition of S_j^1 and the second inequality holds since x is in the core of the assignment game. Equality (6) implies (3), completing the proof of both Claim 3 and Theorem 3.1. \square

The following example illustrates the outcome of the procedure used in the proof of Theorem 3.1 and shows that an extreme point of the core could be generated by several marginal vectors.

Example 3.5 Let $(M \sqcup N; w)$ be the assignment game defined by $M := \{1; 3; 5; 7; 9; 11\}$, $N := \{2; 4; 6; 8; 10; 12\}$, and $w(f_i; j_g) := 1$ if $f_i; j_g$ is an edge in the graph depicted in Figure 3.1 and 0 otherwise. Here, the number in a vertex denotes the corresponding player. The allocation $x = (0; 1; 0; 1; 0; 1; 0; 1; 0; 1; 0; 1)$ is an extreme point of the core of the assignment game $(M \sqcup N; w)$. For both components of the tight graph $G^w(u; v)$, a tight tree is depicted in Figure 1.

The labeling procedure, starting in the vertices 1 and 9, can give the orders

$$\begin{aligned} \circledast &:= (1; 2; 3; 4; 5; 6; 7; 8; 9; 10; 11; 12) \\ - &:= (1; 6; 7; 2; 3; 4; 5; 8; 9; 10; 11; 12) \\ \circ &:= (9; 10; 11; 12; 1; 2; 3; 4; 5; 6; 7; 8) \\ \pm &:= (9; 10; 11; 12; 1; 6; 7; 2; 3; 4; 5; 8) \end{aligned}$$

It is easy to verify that $m^{\circledast}(w) = m^-(w) = m^\circ(w) = m^\pm(w) = x$. \square

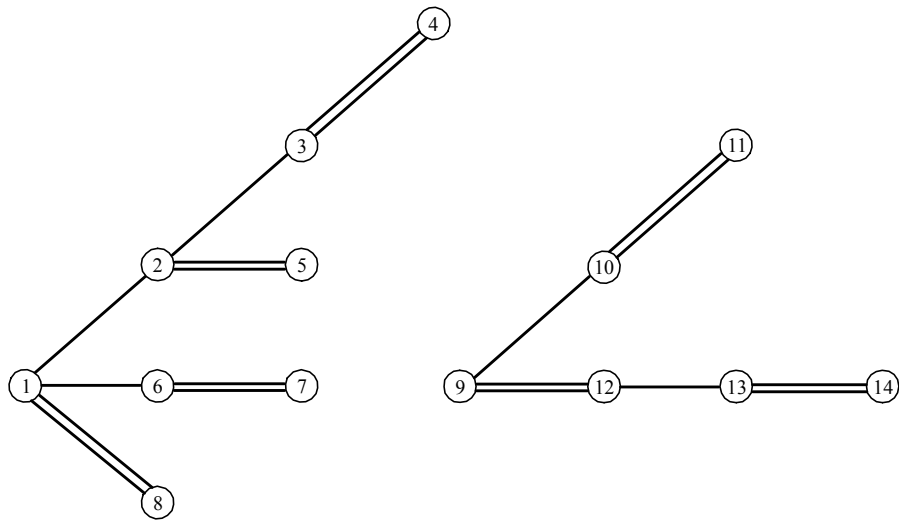


Figure 1: Two tight trees of the components of $G^w(x)$

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